Mixed Mimetic Spectral Element method applied to Darcy's problem

Pedro Pinto Rebelo, Artur Palha and Marc Gerritsma

Abstract We present a discretization for Darcy's problem using the recently developed *Mimetic Spectral Element Method* [19]. The gist lies in the exact discrete representation of integral relations. In this paper, an anisotropic flow through a porous medium is considered and a discretization of a full permeability tensor is presented. The performance of the method is evaluated on standard test problems, converging at the same rate as the best possible approximation.

1 DARCY FLOW

Anisotropic heterogeneous diffusion problems are ubiquitous across different scientific fields, such as, hydrogeology, oil reservoir simulation, plasma physics, biology, etc [10]. Darcy's equation describes a steady pressure-driven flow through a porous medium where fluxes and pressure are linearly related,

$$
\operatorname{div} \frac{\mathbb{K}}{\mu} \operatorname{grad} p = \phi \xrightarrow{\mu=1} \begin{cases} \mathbf{u} - \operatorname{grad} p = 0 & \text{in } \Omega \qquad (1a) \\ \operatorname{div} \mathbf{q} = \phi & \text{in } \Omega \qquad (1b) \\ \mathbf{q} = \mathbb{K}\mathbf{u} & \text{in } \Omega \qquad (1c) \\ \mathbf{q} = \mathbf{q_0} & \text{in } \partial\Omega \qquad (1d) \end{cases}
$$

where $\mathbf{u}$ is the fluid velocity, p the pressure, $\mathbf{q}$ the mass flux and ϕ the prescribed source term. Without loss of generality let the viscosity, $\mu = 1$, and consider a permeability symmetric, positive definite tensor denoted by $\mathbb{K}$.

In a three-dimensional setting are: four types of submanifolds (*points*, *lines*, *surfaces and volumes*); and two orientations (*outer and inner*, as an example see Figure 1). Tessellation divides the physical domain in a set of these geometric objects to

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which we associate discrete variables, i.e. *integral quantities*. Thus, associated with every physical variable is a correspondent geometric object, this symbiotic relation between physics and geometry is the core of *mimetic methods*. Many scholars are aware of this relationship [3, 5, 7, 23].

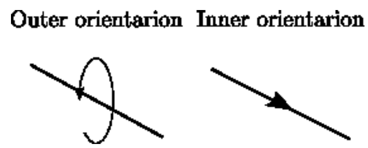

Fig. 1 Consider a line where we can have two types of orientation: *Outer* - around the line; *Inner* - along the line.

Starting from the mass balance equation, (1b),

$$\int_V \operatorname{div} \mathbf{q} \, \mathrm{d}V = \int_{\partial V} \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}S = \int_V \phi \, \mathrm{d}V, \tag{2}$$

it is clear that the *divergence in a volume* is equal to the sum of the *surface integral quantities*, i.e. *oriented fluxes*. Thus, we will associate mass fluxes, $\mathbf{q}$, with quantities that go *through* surfaces. This equation therefore tells us that the right hand side term ϕ is associated to *outer-oriented volumes*.

Similarly, using Newton-Leibniz relation for equation (1a),

$$\int_C \operatorname{grad} p \, \mathrm{d}C = \int_{\partial C} p = p(B) - p(A) = \int_C \mathbf{u} \, \mathrm{d}C, \tag{3}$$

the fluid velocity, $\mathbf{u}$, is represented *along* lines and p is represented by the values in points. From (3) we deduces that u and p are *inner-oriented variables*.

The *constitutive/material relation* relation (1c) is given by,

$$\mathbf{q} = \mathbb{K}\mathbf{u}, \tag{4}$$

which defines how quantities associated to inner-oriented lines relate to quantities associated to outer-oriented surfaces. Whereas equation (2) and (3) can be exactly satisfied on a finite grid the constitutive equation (4) needs to be approximated.

The importance of respecting the geometric nature in physics is discussed in [9]. Figure 2 summarizes the geometric character of the Darcy's problem.

We will denote the space of variables associated to outer-oriented k-dimensional objects by $\Lambda^k(\mathcal{M})$ and the space of variables associated to inner-oriented k-dimensional objects by $\tilde{\Lambda}^k(\mathcal{M})$ as indicated in Figure 2.

In this paper we will make use of the spectral element method described in [9,19], application of these ideas to Stokes' flow see [16–18]; Poisson equation for volume forms [22]; advection equation [21]; derivation of a momentum conservation scheme [24]. Extension to compatible isogeometric methods see [11, 12]. For applications of these ideas in a finite difference setting see Brezzi et al. [4]. In the context

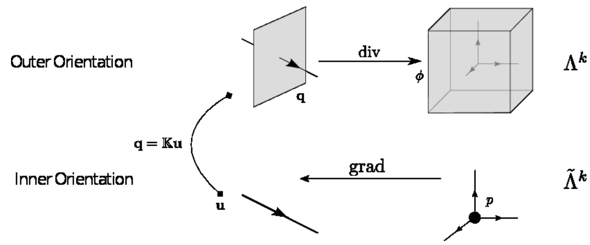

Fig. 2 Darcy's flow problem geometric characterization. Fluxes, $\mathbf{q}$, are associated with outer oriented surfaces; ϕ, is associated with outer oriented volumes; velocity, $\mathbf{u}$, is associated with inner oriented lines; pressure, p, is associated with inner oriented points

of finite element methods Arnold, Falk and Winther [1] proposed a *Finite Element Exterior Calculus*. In a more geometric spirit Desbrun et al. [6] and Hirani [13] developed the *discrete exterior calculus* (DEC). An application of the latter to Darcy flow can be found in [14].

2 DISCRETIZATION OF EQUATIONS

In this section we will describe the discretization by defining the weak formulation. The approached followed here is similar to [2, 15].

For vectors associated with outer-oriented surfaces, $\Lambda^2(\mathcal{M})$, we define the weighted inner product

$$(\mathbf{a},\mathbf{b})_{\mathcal{M},\mathbb{K}} = \int_{\mathcal{M}} \mathbf{a}\mathbb{K}^{-1}\mathbf{b}\,\mathrm{d}V. \tag{5}$$

Furthermore, we define bilinear maps $((\cdot,\cdot))_{\mathcal{M}} : \Lambda^1(\mathcal{M})\times\tilde{\Lambda}^{n-1}(\mathcal{M}) \to \mathbb{R}$

$$((\mathbf{u},\tilde{\mathbf{v}}))_{\mathcal{M}} := \int_{\mathcal{M}} \mathbf{u}\cdot\tilde{\mathbf{v}}\,\mathrm{d}V. \tag{6}$$

and $((\cdot,\cdot))_{\mathcal{M},\mathbb{K}} : \Lambda^k(\mathcal{M})\times\tilde{\Lambda}^{n-k}(\mathcal{M}) \to \mathbb{R}$ given by

$$((\mathbf{u},\tilde{\mathbf{v}}))_{\mathcal{M},\mathbb{K}} := \int_{\mathcal{M}} \mathbf{u}\cdot\mathbb{K}^{-1}\tilde{\mathbf{v}}\,\mathrm{d}V. \tag{7}$$

For $\mathbf{q}\in\Lambda^2(\mathcal{M})$ and $p\in\tilde{\Lambda}^0(\mathcal{M})$ and homogeneous boundary values we have

$$\begin{aligned}((\mathrm{div}\mathbf{q},\ p))_{\mathcal{M}} &= -((\mathbf{q},\mathrm{grad}p))_{\mathcal{M}}\\ &= -\left(\left(\mathbf{q},\mathbb{K}^{-1}\left[\mathbb{K}\ \mathrm{grad}p\right]\right)\right)_{\mathcal{M}}\\ &= \left(\left(\mathbf{q},\mathrm{grad}^*_{\mathbb{K}}\ p\right)\right)_{\mathcal{M},\mathbb{K}}.\end{aligned} \tag{8}$$

It is possible to define a new gradient operator,

$$\mathrm{grad}^{*}_{\mathbb{K}} = -\mathbb{K}\ \mathrm{grad}. \tag{9}$$

Mixed formulation

Starting from (1) and making use of the bilinear maps defined above we have for all vectors $\boldsymbol{\tau} \in \Lambda^{n-1}(\mathcal{M})$ associated to outer-oriented surfaces

$$\begin{aligned}
((\boldsymbol{\tau}, \mathbf{u} - \mathrm{grad} p))_{\mathcal{M}} &= 0 \\
&\Longleftrightarrow \\
((\boldsymbol{\tau}, \mathbb{K}\mathbf{u} - \mathbb{K}\mathrm{grad} p))_{\mathcal{M},\mathbb{K}} &= 0 \\
&\overset{(9)}{\Longleftrightarrow} \\
((\boldsymbol{\tau}, \mathbb{K}\mathbf{u}))_{\mathcal{M},\mathbb{K}} + \left(\boldsymbol{\tau}, \mathrm{grad}^{*}_{\mathbb{K}} p\right)_{\mathcal{M},\mathbb{K}} &= 0 \\
&\overset{(1c)\text{ and }(8)}{\Longleftrightarrow} \\
(\boldsymbol{\tau}, \mathbf{q})_{\mathcal{M},\mathbb{K}} + ((\mathrm{div}\ \boldsymbol{\tau},\ p))_{\mathcal{M}} &= 0
\end{aligned} \tag{10}$$

The constitutive equation is included in the last step by converting the bilinear form to a weighted inner product on $\Lambda^{2}(\mathcal{M})$ as defined in (5). For (1b) we take the bilinear map for the divergence of a vector $\mathbf{q}$ associated with outer-oriented surfaces and an arbitrary scalar function defined in inner-oriented points, $\gamma \in \tilde{\Lambda}^{0}(\mathcal{M})$,

$$((\mathrm{div}\mathbf{q}, \gamma))_{\mathcal{M}} = ((\phi, \gamma))_{\mathcal{M}}. \tag{11}$$

The mixed formulation becomes: Find $(\mathbf{q}, p) \in \left\{\Lambda^{2}(\mathcal{M}) \times \tilde{\Lambda}^{0}(\mathcal{M})\right\}$, given $\phi \in \Lambda^{3}(\mathcal{M})$, for all $(\boldsymbol{\tau}, \gamma) \in \left\{\Lambda^{2}(\mathcal{M}) \times \tilde{\Lambda}^{0}(\mathcal{M})\right\}$ such that,

$$(\tau, \mathbf{q})_{\mathcal{M},\mathbb{K}} + ((\mathrm{div}\ \boldsymbol{\tau},\ p))_{\mathcal{M}} = 0 \tag{12}$$

$$((\mathrm{div}\mathbf{q}, \gamma))_{\mathcal{M}} = ((\phi, \gamma))_{\mathcal{M}}\ . \tag{13}$$

2.1 Basis functions

For the high order representation we use Lagrange, $l_i(\xi)$, and edge functions, $e_i(\xi)$. Lagrange polynomials interpolate nodal values. The edge functions, derived by Ger-

ritsma [8] are constructed such that when integrating over a line segment it gives one for the corresponding element and zero for any other line segment,

$$l_i\left(\xi_j\right) = \delta_{i,j} \qquad \int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \delta_{i,j}. \tag{14}$$

The relation between the Lagrange and the edge functions is given by,

$$e_i(\xi) = \epsilon_i(\xi)\, d\xi, \quad \text{with} \quad \epsilon_i(\xi) = -\sum_{k=0}^{i-1} \frac{dl_k}{d\xi}. \tag{15}$$

Note that this definition implies

$$\frac{dl_i}{d\xi} = e_i(\xi) - e_{i+1}(\xi)\ . \tag{16}$$

Extension to the multidimensional is obtained by means of tensor products. For more details see [19].

2.2 Mimetic discretization in 2D

Expansion of unknowns in $\mathbb{R}^2$

Let $\mathbf{q} \in \Lambda^1(\mathcal{M})$ be expanded as,

$$\mathbf{q}_h = \begin{bmatrix} \sum_{i=0}^{N} \sum_{j=1}^{N} q^x_{i,j} l_i(\xi)\, e_j(\eta) \\ \sum_{i=1}^{N} \sum_{j=0}^{N} q^y_{i,j} e_i(\xi)\, l_j(\eta) \end{bmatrix}, \tag{17}$$

and the pressure, $p_h \in \tilde{\Lambda}^0(\mathcal{M})$ as,

$$p_h = \sum_{i=1}^{N} \sum_{j=1}^{N} p_{i,j} \epsilon_i(\xi)\, \epsilon_j(\eta). \tag{18}$$

Discrete divergence in $\mathbb{R}^2$

The divergence of $\mathbf{q}_h$ is then given by

$$\operatorname{div} \mathbf{q}_h = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(q^x_{i,j} - q^x_{i-1,j} + q^y_{i,j} - q^y_{i,j-1} \right) e_i(\xi)\, e_j(\eta)\ , \tag{19}$$

where we repeatedly used (16). The scalar $\phi \in \Lambda^2(\mathcal{M})$ associated with outer-oriented volumes is expanded as

$$\phi_h = \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_{i,j} e_i(\xi) e_j(\eta) \ . \tag{20}$$

Equating (19) and (20) yields

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \phi_{i,j} \cancel{e_i(\xi) e_j(\eta)} = \sum_{i=1}^{N}\sum_{j=1}^{N} \left(q^x_{i,j} - q^x_{i-1,j} + q^y_{i,j} - q^y_{i,j-1}\right) \cancel{e_i(\xi) e_j(\eta)} \tag{21}$$

$$[\phi] = \mathsf{E}^{(2,1)}[\mathbf{q}] \, . \tag{22}$$

We see that the basis functions cancel from this relation. The matrix $\mathsf{E}^{(2,1)}$ relates the fluxes $q^x_{i,j}$ and $q^y_{i,j}$ to the volume integral $\phi_{i,j}$, as depicted in Figure 3. This fully discrete equation is a restatement of the integral relation (2). The matrix $\mathsf{E}^{(2,1)}$ only contains the values -1, 0 and 1 and is fully determined by the grid, see [19]. This is an incidence matrix showing the topological nature of the discrete divergence.

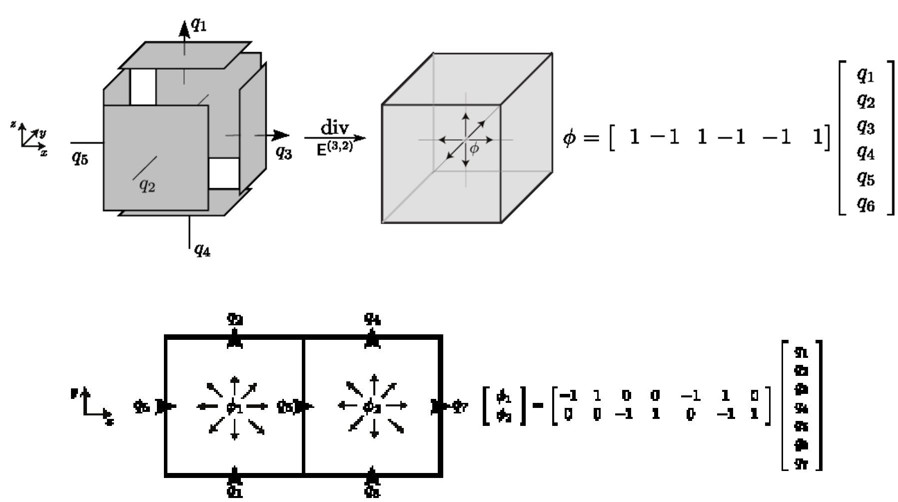

Fig. 3 Discrete representation of the action of the divergence in $\mathbb{R}^3$ and $\mathbb{R}^2$.

If we insert the expansions of our unknowns in (12) and (13) we obtain in $\mathbb{R}^n$ the saddle point problem given by,

$$\begin{bmatrix} M_{\mathbb{K}}^{(n-1)} & \left(\mathsf{E}^{(n,n-1)}\right)^T M^{(n)} \\ M^{(n)}\mathsf{E}^{(n,n-1)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ M^{(n)}\phi \end{bmatrix}, \tag{23}$$

where $M^{(k)}$ is the symmetric mass matrix obtain from the bilinear pairing between variables associated with outer-orientation and inner-orientation, (6), $M_{\mathbb{K}}^{(n-1)}$ is the mass matrix obtained from the weighted inner product (5) and $\mathsf{E}^{(n,n-1)}$ the incidence matrix which relates fluxes over surfaces to volumes. The resulting system (23) is symmetric.

The pressure which is represented on an inner-oriented grid (which is not explicitly constructed in this single grid approach) is pre-multiplied by $M^{(n)}$ to represent it on the outer-oriented grid.

3 NUMERICAL RESULTS

The method derived in this paper respects the geometric nature of the problem. However, it is crucial to verify the numerical benefits of this approach. This section presents $hp-$convergence studies for anisotropic permeability.

3.1 Manufactured solution - Anisotropic permeability

The first test case assesses the convergence for $h-$ and $p-$refinement of the mixed mimetic spectral element method applied to the Darcy model. This is a benchmark problem presented in [15]. The problem is defined on a unit square, $\Omega = [-1,1]^2$, with Cartesian coordinates with permeability given by,

$$\mathbb{K} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{24}$$

and the right hand side, $\phi \in L^2(\mathcal{M})$ given by,

$$\phi^{(2)} = 2\left(1 + x^2 + xy + y^2\right)e^{xy}\,\mathrm{d}x\mathrm{d}y. \tag{25}$$

This results in an exact solution for pressure $p \in \tilde{\Lambda}^0(\mathcal{M})$ given by,

$$p^{(0)} = e^{xy} \tag{26}$$

Figure 4 shows the $h-$ and $p-$convergence for the pressure in straight mesh. For the $h-$convergence the expected rate of convergence is of $(p+1)$, where p is the polynomial degree. The solid line for the interpolation error in the p-convergence plot

is the L^2-error from interpolating the exact solution, the solution converges exponentially. Both the numerical solution and the interpolated exact solution converge exponentially.

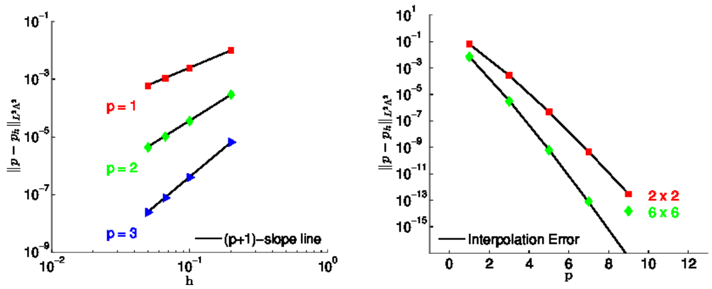

Fig. 4 Plots of the $h-$ and $p-$convergence for anisotropic permeability given in (24).

3.2 Layered medium

A classical benchmark for Darcy flow codes is the piecewise constant permeability in a square [20]. Such a medium is called *layered* medium.

$$\mathbb{K} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \quad \alpha = \begin{cases} 0.3 & \text{if } y \leq \frac{1}{3} \\ 0.7 & \text{if } \frac{1}{3} < y \leq \frac{2}{3} \\ 0.5 & \text{if } y > \frac{2}{3} \end{cases} \tag{27}$$

The fluid comes into the domain from the left to the right. Since the pressure depends linearly on x, horizontal constant velocity is expected in each layer, Figure 5.

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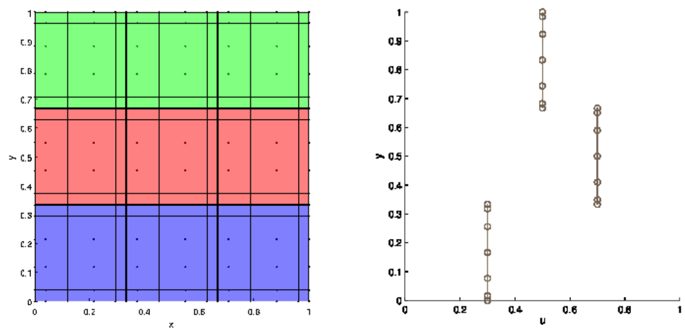

Fig. 5 Layered medium with 3 different permeabilities, α ranges top to bottom from 0.5, 0.7 and 0.3. The left figure shows a velocity profile at an arbitrary x. Note the numerical solution in the Gauss-Lobatto nodes.

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First of all the authors would like to acknowledge the reviewers for their feedback.

1 Reviewer 1 :: ICO12-2-12.R1.pdf

- An extra figure was added for clarity of what is meant with two orientations (Figure 1). The reader is also directed to the paper on this issue, *The geometric basis of numerical methods*, where a careful analysis of geometry is given.
- Formula (14) is corrected.
- Formula (27) is corrected since pressure belongs to $\tilde{\Lambda}^0(\mathcal{M})$ the term $dxdy$ should vanish.
- Reference 1 was completed together with some other references that were recently published.

English was carefully reviewed and includes all the comments.

2 Reviewer 2 :: ICO12-2-12.R2.pdf

- A reference to the research areas and the FVCA6 benchmark is added to the first sentence.
- Citations referring to the work of Bossavit and Tonti were added.
- Citations about Finite Element Calculus, Arnold et al; Mimetic Finite Difference, Brezzi et al; Analysis of Compatible Discrete Operator Schemes for Elliptic Problems on Polyhedral Meshes, Bonelle et al.; DEC on Darcy, Hirani et al.
- In our formulation we avoid the term *scalar product* because of entries in this bilinear form come from different spaces, inner and outer oriented variables associated with different geometric objects. This is the main idea in this paper, therefore we prefer to use *bi-linear* form.
- The term adjoint was removed.

- The sentence was changed according to the reviewer suggestion.

All the minor reviews were taken into account.